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Decision-Making Models Using Type-2 Fuzzy Sets: Formulation, Aggregation and Ranking of Alternatives under Higher-Order Uncertainty

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Abstract

Background: Decisions in engineering, management and public policy are routinely made from linguistic judgements such as “high” or “moderate”. Ordinary (type-1) fuzzy sets represent such judgements with a single, exact membership function, which hides the fact that different experts, and the same expert at different times, attach different meanings to the same word. Type-2 fuzzy sets model this second layer of uncertainty explicitly. **Methods:** This article reviews the mathematical foundations of general and interval type-2 fuzzy sets, including the footprint of uncertainty, embedded sets, set operations and type reduction by the Karnik–Mendel and Nie–Tan methods. It then formulates a general multi-criteria decision-making model in which ratings and weights are interval type-2 fuzzy numbers, specifies the aggregation and ranking steps, and works through a numerical example with three alternatives and three criteria. **Results:** In the example the alternatives are ranked $A_1 \succ A_3 \succ A_2$ with defuzzified scores 7.071, 6.500 and 5.839. The distance between the upper and lower centroids gives a usable index of how much linguistic uncertainty sits behind each score. **Conclusion:** Interval type-2 models keep the computational cost close to that of type-1 models while carrying information about disagreement that type-1 models discard, which makes them a practical default for group decisions based on words.

Keywords: type-2 fuzzy sets; interval type-2 fuzzy sets; multi-criteria decision-making; footprint of uncertainty; type reduction; Karnik–Mendel algorithm

1. Introduction

Most real decisions are taken with incomplete information. A district planner choosing between irrigation projects, a hospital selecting a supplier or an engineer comparing materials must weigh several criteria at once, and much of the evidence arrives as expert opinion expressed in words. Zadeh’s fuzzy set theory [1] gave a rigorous way to handle such vague terms by allowing an element to belong to a set to a degree between 0 and 1, and Bellman and Zadeh [2] soon showed how fuzzy goals and constraints could be combined into a decision. Since then fuzzy multi-criteria decision-making (MCDM) has grown into a large field, with fuzzy versions of the analytic hierarchy process, TOPSIS, VIKOR and many other methods [3,4].

A type-1 fuzzy set, however, still asks the modeller for something exact: a single crisp membership grade for every element. If ten experts are asked where “high cost” begins, they give ten different answers, and a single membership function cannot record that spread. Zadeh himself anticipated this problem in 1975 when he introduced fuzzy sets whose membership grades are themselves fuzzy, which he called type-2 fuzzy sets [5]. Mendel and John [6] later gave the now-standard formal definition and the notion of a footprint of uncertainty, and the theory became practical once efficient type-reduction algorithms were found [7,8].

Type-2 models are now used in control, forecasting and pattern recognition, and a steady stream of work has extended them to decision-making [9–12]. Chen and Lee [9], for example, built an interval type-2 TOPSIS method, and Mendel and Wu [10] developed the linguistic weighted average for computing with words. Yet for students and practitioners entering the area the literature is scattered, notation differs between papers, and the step from definitions to a working decision model is rarely shown in one place.

This article therefore has three aims. First, it sets out the core definitions of type-2 and interval type-2 fuzzy sets in a consistent notation. Second, it formulates a general decision model in which ratings and weights are interval type-2 fuzzy numbers and states each computational step. Third, it demonstrates the model on a fully worked example and discusses when the extra modelling effort is worthwhile.

2. Mathematical Preliminaries

2.1. Type-1 Fuzzy Sets

Let X be a universe of discourse. A type-1 fuzzy set A in X is characterised by a membership function $\mu_A : X \to [0,1]$, and is written $A = \{(x, \mu_A(x)) : x \in X\}$. A trapezoidal

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fuzzy number with parameters (a,b,c,d), $a \le b \le c \le d$, has membership rising linearly from a to b, equal to 1 between b and c, and falling linearly from c to d. Triangular numbers are the special case b = c.

2.2. General Type-2 Fuzzy Sets

A type-2 fuzzy set $\tilde{A}$ in X is characterised by a type-2 membership function $\mu_{\tilde{A}}(x,u)$, where $x \in X$ and $u \in J_x \subseteq [0,1]$ [6]:

$$\tilde{A} = \{((x,u), \mu_{\tilde{A}}(x,u)) : x \in X,\ u \in J_x\},\ 0 \le \mu_{\tilde{A}}(x,u) \le 1 \quad (1)$$

For each x, the set J_x is the primary membership of x, and the function $u \mapsto \mu_{\tilde{A}}(x,u)$ is the secondary membership function, a type-1 fuzzy set on [0,1]. In words, a type-2 set does not give one grade for x; it gives a fuzzy set of possible grades. This is what allows the model to represent disagreement about the meaning of a word.

2.3. Interval Type-2 Fuzzy Sets and the Footprint of Uncertainty

General type-2 sets are expensive to compute with, because every point carries a full secondary function. When all secondary grades equal 1 the set is called an interval type-2 fuzzy set (IT2 FS). The union of all primary memberships is the footprint of uncertainty (FOU):

$$\mathrm{FOU}(\tilde{A}) = \cup_{x \in X} J_x = \{(x,u) : u \in [\underline{\mu}_{\tilde{A}}(x), \bar{\mu}_{\tilde{A}}(x)]\} \quad (2)$$

The FOU is bounded above by the upper membership function (UMF) $\bar{\mu}_{\tilde{A}}$ and below by the lower membership function (LMF) $\underline{\mu}_{\tilde{A}}$, both type-1 functions. An IT2 FS is therefore completely described by two type-1 sets, which is why its computational cost stays modest. Any type-1 function lying entirely within the FOU is called an embedded set, and by the wavy-slice representation theorem of Mendel and John [6] the IT2 FS equals the union of all its embedded sets. The width of the FOU is a direct picture of the uncertainty about the word: a narrow FOU means experts largely agree, a wide one means they do not.

In decision applications it is common to use trapezoidal IT2 fuzzy numbers, written

$$\tilde{A} = ((a_1,a_2,a_3,a_4;1), (b_1,b_2,b_3,b_4;h)) \quad (3)$$

where the first trapezoid is the UMF with height 1 and the second is the LMF with height $h \in (0,1]$, with $a_1 \le b_1$ and $b_4 \le a_4$ so that the LMF lies inside the UMF [9].

2.4. Set Operations and Arithmetic

For IT2 FSs, union and intersection reduce to operations on the bounding functions. With the maximum t-conorm and minimum t-norm, the union of $\tilde{A}$ and $\tilde{B}$ has UMF $\max(\bar{\mu}_{\tilde{A}}, \bar{\mu}_{\tilde{B}})$ and LMF $\max(\underline{\mu}_{\tilde{A}}, \underline{\mu}_{\tilde{B}})$, and the intersection uses min in both places; the complement has UMF $1 - \underline{\mu}_{\tilde{A}}$ and LMF $1 - \bar{\mu}_{\tilde{A}}$. For trapezoidal IT2 numbers, addition and multiplication by a positive scalar k are carried out parameter-wise on the UMF and the LMF, the LMF height being taken as the minimum of the operands' heights [9]:

$$\tilde{A} \oplus \tilde{B} = ((a_i + a'_i; 1), (b_i + b'_i; \min(h,h'))) \quad (4a)$$

$$k\tilde{A} = ((ka_i; 1), (kb_i; h)),\ k > 0 \quad (4b)$$

These rules are approximations of the exact extension-principle results, but they preserve the trapezoidal form, are easy to compute and are standard in the IT2 MCDM literature.

2.5. Type Reduction and Defuzzification

To compare alternatives, an IT2 FS must eventually be reduced to a number. The centroid of an IT2 FS is the set of centroids of all its embedded type-1 sets, which is an interval $[c_l, c_r]$ [7]. Karnik and Mendel [7] showed that, on a discretised domain $x_1 < \ldots < x_N$, the end points are attained by switching from the UMF to the LMF at a single point:

$$c_l = \min_k \left(\Sigma_{i \le k}\, x_i \bar{\mu}_i + \Sigma_{i > k}\, x_i \underline{\mu}_i\right) / \left(\Sigma_{i \le k}\, \bar{\mu}_i + \Sigma_{i > k}\, \underline{\mu}_i\right) \quad (5)$$

with c_r obtained symmetrically by using the LMF on the left and the UMF on the right. The iterative Karnik–Mendel (KM) algorithm finds the switch points in a few iterations [8]. The defuzzified value is then $(c_l + c_r)/2$. A popular closed-form alternative is the Nie–Tan method [13], which averages the bounding functions before computing an ordinary centroid:

$$y_{NT} = \Sigma_i\, x_i\, [\bar{\mu}(x_i) + \underline{\mu}(x_i)] \,/\, \Sigma_i\, [\bar{\mu}(x_i) + \underline{\mu}(x_i)] \quad (6)$$

The Nie–Tan value closely approximates the KM midpoint and requires no iteration, which makes it attractive when many alternatives must be ranked.

3. Proposed Decision-Making Model

3.1. Problem Setting

Consider m alternatives $A_1, \ldots, A_m$ evaluated against n criteria $C_1, \ldots, C_n$ by a panel of K decision-makers. Each decision-maker gives a linguistic rating of every alternative on every criterion and a linguistic or crisp importance for every criterion. The goal is a complete ranking of the alternatives together with an indication of how robust that ranking is to linguistic uncertainty.

3.2. Model Steps

Step 1 — Construct the linguistic codebook. A set of terms such as Very Low (VL), Low (L), Medium (M), High (H) and Very High (VH) is defined on a common scale [0,10]. Each term is assigned an IT2 trapezoidal number of the form (3). Ideally the FOUs are estimated from interval data collected from experts, for example by the interval approach of Liu and Mendel [14]; otherwise they are fixed by the analyst so that neighbouring terms overlap.

Step 2 — Aggregate the panel. For each alternative i and criterion j, the K ratings $\tilde{r}_{ij}^{(k)}$ are combined by the arithmetic mean using (4): $\tilde{r}_{ij} = (1/K) \bigoplus_k \tilde{r}_{ij}^{(k)}$. Criterion weights are aggregated in the same way.

Step 3 — Normalise. If all criteria share the same scale and direction, no normalisation is needed. For cost criteria the scale is reversed by replacing each parameter p with 10 − p and reordering, so that larger values are always better.

Step 4 — Compute the weighted score. With crisp normalised weights w_j ($\Sigma w_j = 1$) the overall IT2 score of alternative i is

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$$\tilde{S}_i = \bigoplus_{j=1..n} w_j \tilde{r}_{ij} \quad (7)$$

When the weights are themselves IT2 numbers, the linguistic weighted average of Mendel and Wu [10] is used in place of (7); it computes the UMF and LMF of the result by α-cut-wise KM procedures.

Step 5 — Rank. Each $\tilde{S}_i$ is defuzzified, by the KM midpoint or by (6), and the alternatives are ordered by decreasing value. The distance between the centroid of the UMF and the centroid of the LMF, or the width $c_r - c_l$, is reported alongside the score as an uncertainty index U_i.

Step 6 — Check robustness. The ranking is recomputed after perturbing the weights by ±10% and, where available, by deleting one decision-maker at a time. A ranking that survives these checks can be presented with confidence; one that does not signals that more information is needed.

4. Numerical Illustration

4.1. Data

A small public-works department must choose among three solar water-pumping suppliers A_1, A_2 and A_3 using three benefit criteria: C_1 technical quality (weight 0.40), C_2 after-sales service (0.35) and C_3 cost-effectiveness (0.25). The linguistic codebook is given in Table 1; all LMFs have height h = 0.8. The aggregated panel ratings are A_1 = (H, M, VH), A_2 = (VH, L, M) and A_3 = (M, H, H) on (C_1, C_2, C_3).

Table 1. Linguistic terms and their interval type-2 trapezoidal representations (UMF height 1, LMF height 0.8).

Term	UMF (a_1,a_2,a_3,a_4)	LMF (b_1,b_2,b_3,b_4)
VL	(0, 0, 1, 2)	(0.5, 0.5, 0.75, 1.5)
L	(1, 2, 3, 4)	(1.5, 2.25, 2.75, 3.5)
M	(3, 4, 6, 7)	(3.5, 4.5, 5.5, 6.5)
H	(6, 7, 8, 9)	(6.5, 7.25, 7.75, 8.5)
VH	(8, 9, 10, 10)	(8.5, 9.25, 10, 10)

4.2. Computation

Applying (7) parameter-wise gives the overall scores. For A_1, the UMF is 0.40(6,7,8,9) + 0.35(3,4,6,7) + 0.25(8,9,10,10) = (5.45, 6.45, 7.80, 8.55), and the LMF is (5.95, 6.79, 7.52, 8.18) with height 0.8. The other alternatives are obtained in the same way. Each score was then evaluated on a grid of 10,001 points over [0,10], the UMF and LMF centroids were computed, and the Nie–Tan value (6) was used for ranking. Table 2 reports the results.

Table 2. Aggregated IT2 scores, centroids and final ranking.

Alt.	UMF of $\tilde{S}_i$	LMF of $\tilde{S}_i$	y_{NT}	Rank
A_1	(5.45, 6.45, 7.80, 8.55)	(5.95, 6.79, 7.52, 8.18)	7.071	1
A_2	(4.30, 5.30, 6.55, 7.15)	(4.80, 5.61, 6.34, 6.85)	5.839	3
A_3	(4.80, 5.80, 7.20, 8.20)	(5.30, 6.15, 6.85, 7.70)	6.500	2

UMF and LMF centroids: A_1 7.054 and 7.102; A_2 5.812 and 5.888; A_3 6.500 and 6.500.

4.3. Interpretation

The final ranking is $A_1 \succ A_3 \succ A_2$. Supplier A_1 wins because it combines a high technical rating with a very high cost-effectiveness rating, while A_2's very high technical quality is offset by weak after-sales service, the second most important criterion. The gap between the first and second alternatives (0.571) is much larger than the gap between the UMF and LMF centroids of any alternative (at most 0.076), so linguistic uncertainty in the codebook is not large enough to reverse the order. A_3 has symmetric ratings and its two centroids coincide, which shows that the uncertainty index measures asymmetry in the FOU rather than its mere width; for a full width measure the KM interval $[c_l, c_r]$ should be reported. Perturbing the weights by ±10% one at a time leaves the ranking unchanged in this example, because A_1 is rated at least as well as A_3 on two of the three criteria.

5. Discussion

The example shows the main practical attraction of interval type-2 models: they reuse the familiar structure of a weighted-sum decision while carrying an extra layer of information. A type-1 model built from the UMFs alone would produce almost the same ranking here, but it would offer no way of telling a decision-maker whether the ranking rests on firm or fragile judgements. In the type-2 model that information is a by-product of the same computation.

Three choices in the model deserve comment. First, the arithmetic of (4) is an approximation. Exact IT2 arithmetic via the extension principle produces non-trapezoidal results, and for large or nonlinear models the α-cut-based linguistic weighted average [10] is preferable. Second, the choice of type-reduction method matters less for ranking than for the width of the uncertainty interval. Studies comparing KM, enhanced KM and closed-form methods report that rankings rarely differ, while computing time can fall by an order of magnitude with closed-form methods [8,13]. Third, the codebook is the weakest link. FOUs fixed arbitrarily by the analyst carry little real meaning; they should be estimated from survey data on how respondents interpret each word [14].

Type-2 models are not always worth the extra effort. When decision-makers agree closely, or when ratings are numerical measurements rather than words, a type-1 or even crisp model is adequate. The type-2 approach pays off in group decisions with heterogeneous experts, in problems where the same word is used across cultures or languages, and in settings where justifying the robustness of a ranking matters as much as the ranking itself. Extensions such as hesitant, intuitionistic and neutrosophic fuzzy sets address related concerns [15,16], but IT2 FSs have the advantage of a mature computational theory and a clear interpretation of the FOU.

The study has limitations. The worked example is illustrative and small, and the ratings were set by the authors rather than collected from a real panel. General type-2 sets, which can represent uncertainty more finely

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through non-uniform secondary grades, were reviewed but not applied. Future work will estimate codebooks from field data, compare IT2 TOPSIS, VIKOR and weighted-sum models on the same problem, and study zSlice-based general type-2 decision models [17].

6. Conclusions

Type-2 fuzzy sets extend fuzzy decision-making by admitting that the meaning of a word is itself uncertain. Interval type-2 sets capture this through a footprint of uncertainty bounded by two ordinary membership functions, so they can be manipulated with tools that practitioners already know. This article has brought the necessary definitions together in a consistent notation, stated a six-step decision model with aggregation, ranking and robustness checks, and demonstrated it on a worked example in which the ranking $A_1 \succ A_3 \succ A_2$ was accompanied by a measure of the uncertainty behind each score. The approach is a sound default for group decisions based on linguistic judgements, provided the codebook is grounded in data.

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