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Continuous Mappings and Topological Properties: A Systematic Study of Invariance, Preservation and Counterexamples

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Abstract

Background: Continuous mappings are the structure-preserving maps of topology, and much of the subject consists of asking which properties of a space survive when the space is mapped continuously onto another. The answers are scattered across textbooks, and the conditions under which preservation fails are often left implicit. **Methods:** This article gives a self-contained account of continuity in topological spaces through its equivalent characterisations and then examines, property by property, whether compactness, connectedness, path-connectedness, separability, the countability axioms and the separation axioms are preserved by continuous maps, by continuous surjections, by open or closed maps and by homeomorphisms. Each positive result is proved and each negative result is supported by an explicit counterexample. **Results:** Compactness, connectedness, path-connectedness, separability and the Lindelöf property are preserved under continuous surjections. The Hausdorff and other separation axioms, first and second countability and metrisability are not, although they are preserved under homeomorphisms and, in several cases, under maps that are also open or closed. The results are collected in a single preservation table. **Conclusion:** Distinguishing between continuous, open, closed and bijective maps clarifies exactly which hypotheses each preservation theorem needs, and provides a compact reference for teaching and for research in general topology.

Keywords: continuous mapping; homeomorphism; topological invariant; compactness; connectedness; separation axioms; quotient map

1. Introduction

Topology studies the properties of spaces that do not depend on distance or size, only on the way points are grouped together. The central tool for comparing two topological spaces is the continuous mapping. In elementary analysis a function $f : \mathbb{R} \to \mathbb{R}$ is continuous when small changes in x produce small changes in f(x); in topology this idea is freed from the notion of distance and expressed purely in terms of open sets [1,2]. A map is continuous when the inverse image of every open set is open.

Once continuity is defined, a natural programme follows. If $f : X \to Y$ is continuous and X has some property P, must the image f(X) have P as well? If f is a bijection whose inverse is also continuous, a homeomorphism, then X and Y are topologically indistinguishable, and any property defined purely in terms of open sets is shared by both. Such properties are called topological invariants. But many properties survive under much weaker maps, and some fail even for very well-behaved ones. Knowing which is which is essential when proving that two spaces are not homeomorphic, when building quotient spaces, and when transferring results from one space to another [3,4].

The classical results — that continuous images of compact sets are compact and of connected sets are connected — appear in every textbook. Less attention is usually given to the failures, such as the fact that a continuous image of a Hausdorff space need not be Hausdorff, or that a continuous bijection need not be a homeomorphism. Students often meet these only as scattered exercises.

This article gathers the main results in one place. Section 2 sets out the definitions and equivalent forms of continuity. Section 3 proves the preservation theorems for compactness, connectedness and related properties. Section 4 considers properties that are not preserved and gives counterexamples. Section 5 studies open, closed and quotient maps and summarises all results in a preservation table. Section 6 discusses implications and Section 7 concludes.

2. Preliminaries

2.1. Topological Spaces

A topology on a set X is a family τ of subsets of X, called open sets, that contains $\emptyset$ and X and is closed under arbitrary unions and finite intersections. A set is closed if its complement is open. A subfamily $\mathcal{B} \subseteq \tau$ is a base if every open set is a union of members of $\mathcal{B}$. For $A \subseteq X$, the

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closure $\bar{A}$ is the smallest closed set containing A, and the interior A° the largest open set contained in A [1].

2.2. Continuity and Its Characterisations

Definition 1. Let (X,τ_X) and (Y,τ_Y) be topological spaces. A function $f : X \to Y$ is continuous if $f^{-1}(V) \in \tau_X$ for every $V \in \tau_Y$.

Theorem 1. For $f : X \to Y$ the following are equivalent: (i) f is continuous; (ii) $f^{-1}(F)$ is closed in X for every closed $F \subseteq Y$; (iii) $f^{-1}(B)$ is open for every member B of a base of Y; (iv) $f(\bar{A}) \subseteq \mathrm{cl}(f(A))$ for every $A \subseteq X$; (v) for every $x \in X$ and every neighbourhood V of f(x) there is a neighbourhood U of x with $f(U) \subseteq V$.

Proof. (i)⇔(ii) follows from $f^{-1}(Y\setminus F) = X\setminus f^{-1}(F)$. (i)⇔(iii) holds because every open set is a union of basic sets and inverse images preserve unions. For (ii)⇒(iv), the set $f^{-1}(\mathrm{cl}\, f(A))$ is closed and contains A, hence contains $\bar{A}$. For (iv)⇒(ii), let F be closed and $A = f^{-1}(F)$; then $f(\bar{A}) \subseteq \mathrm{cl}\, f(A) \subseteq F$, so $\bar{A} \subseteq A$ and A is closed. Finally (i)⇒(v) by taking $U = f^{-1}(V)$, and (v)⇒(i) because each point of $f^{-1}(V)$ then has an open neighbourhood inside $f^{-1}(V)$. ■

In metric spaces condition (v) reduces to the familiar ε–δ definition, and in first-countable spaces continuity is equivalent to sequential continuity: $x_n \to x$ implies $f(x_n) \to f(x)$ [2].

2.3. Special Classes of Maps

A map f is open (closed) if f(U) is open (closed) for every open (closed) $U \subseteq X$. A homeomorphism is a continuous bijection whose inverse is continuous; equivalently, a continuous open bijection. A continuous surjection f is a quotient map if $V \subseteq Y$ is open exactly when $f^{-1}(V)$ is open in X. Every continuous open or closed surjection is a quotient map [1].

3. Properties Preserved by Continuous Mappings

3.1. Compactness

A space is compact if every open cover has a finite subcover.

Theorem 2. If $f : X \to Y$ is continuous and X is compact, then f(X) is compact.

Proof. Let $\{V_\alpha\}$ be a cover of f(X) by open sets of Y. Then $\{f^{-1}(V_\alpha)\}$ is an open cover of X, which has a finite subcover $f^{-1}(V_1), \ldots, f^{-1}(V_k)$. Every point of f(X) is f(x) for some x in one of these sets, so $V_1, \ldots, V_k$ cover f(X). ■

Two classical consequences follow. First, a continuous real-valued function on a compact space is bounded and attains its maximum and minimum, because a compact subset of $\mathbb{R}$ is closed and bounded (Heine–Borel). This is the topological core of the extreme value theorem. Second:

Theorem 3. A continuous bijection f from a compact space X onto a Hausdorff space Y is a homeomorphism.

Proof. It suffices to show f is closed. If $F \subseteq X$ is closed, F is compact as a closed subset of a compact space; f(F) is compact by Theorem 2; and a compact subset of a Hausdorff space is closed. ■

The same argument shows that continuous images of countably compact, sequentially compact and Lindelöf spaces are again countably compact, sequentially compact and Lindelöf respectively [3].

3.2. Connectedness and Path-Connectedness

A space is connected if it is not the union of two disjoint non-empty open sets.

Theorem 4. The continuous image of a connected space is connected.

Proof. If $f(X) = U \cup V$ with U, V disjoint, non-empty and open in f(X), then $X = f^{-1}(U) \cup f^{-1}(V)$ is a separation of X, a contradiction. ■

Since the connected subsets of $\mathbb{R}$ are exactly the intervals, Theorem 4 yields the intermediate value theorem: a continuous $f : [a,b] \to \mathbb{R}$ takes every value between f(a) and f(b). A space is path-connected if any two points are joined by a continuous path $\gamma : [0,1] \to X$. If f is continuous and γ joins x to x′, then $f\circ\gamma$ joins f(x) to f(x′), so path-connectedness is also preserved. The number of connected components and of path components cannot increase under continuous surjections, which provides a simple invariant: $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^2$, because deleting one point disconnects $\mathbb{R}$ but not $\mathbb{R}^2$.

3.3. Separability and Density

A space is separable if it has a countable dense subset D. If f is continuous and surjective, then by Theorem 1(iv) $Y = f(X) = f(\bar{D}) \subseteq \mathrm{cl}\, f(D)$, so f(D) is a countable dense subset of Y. Separability is therefore preserved by continuous surjections.

4. Properties Not Preserved by Continuous Mappings

4.1. Separation Axioms

A space is Hausdorff (T_2) if distinct points have disjoint neighbourhoods. Hausdorffness is not preserved by continuous surjections. Let $X = \mathbb{R}$ with the usual topology, $Y = \mathbb{R}$ with the indiscrete topology $\{\emptyset, \mathbb{R}\}$, and f the identity. Every map into an indiscrete space is continuous, so f is a continuous bijection, but Y is not even T_0. The same example shows that regularity, normality and complete regularity are not preserved. It also shows that the inverse of a continuous bijection need not be continuous.

A less trivial example is the quotient of $\mathbb{R}$ obtained by identifying all rationals to a single point; the quotient map is continuous and surjective, but the image is not Hausdorff because the identified point cannot be separated from any irrational.

4.2. Countability Axioms and Metrisability

Let $X = \mathbb{R}$ with the discrete topology, which is metrisable and first countable, and let Y be $\mathbb{R}$ with the cofinite topology; the identity is continuous because every cofinite set is open in X. Y is not first countable, since a countable family of cofinite sets has countable union of complements and cannot form a local base at any point of the uncountable set $\mathbb{R}$. Hence first countability and

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metrisability are not preserved. Second countability also fails under continuous surjections, and even under closed quotient maps. Collapsing the integers in $\mathbb{R}$ to a single point gives a quotient space $\mathbb{R}/\mathbb{Z}$, a countable wedge of intervals, in which the collapsed point has no countable local base, so the image of a second-countable metric space is not even first countable [3].

4.3. Openness and Closedness of Images

Continuity says nothing about images of open or closed sets. The map $f(x) = x^2$ from $\mathbb{R}$ to $\mathbb{R}$ sends the open interval $(-1,1)$ to $[0,1)$, which is not open. The projection $p(x,y) = x$ from $\mathbb{R}^2$ to $\mathbb{R}$ sends the closed hyperbola $\{xy = 1\}$ to $\mathbb{R}\setminus\{0\}$, which is not closed. Hence the notions of open and closed maps are genuinely independent of continuity, and of each other.

5. Open, Closed and Quotient Maps

Adding openness or closedness restores several properties. A continuous open surjection preserves first and second countability, since images of a countable base form a countable base. A continuous closed surjection preserves normality, and a perfect map (continuous, closed, surjective, with compact fibres) preserves the Hausdorff property, regularity, local compactness, second countability and metrisability [3,5]. Homeomorphisms preserve every property defined from the topology alone. Table 1 summarises the results of Sections 3–5.

Table 1. Preservation of topological properties under classes of mappings (✓ preserved; ✗ not preserved in general).

Property	Cont. surj.	Cont. open surj.	Perfect map	Homeo.
Compact	✓	✓	✓	✓
Connected	✓	✓	✓	✓
Path-connected	✓	✓	✓	✓
Separable	✓	✓	✓	✓
Lindelöf	✓	✓	✓	✓
First countable	✗	✓	✗	✓
Second countable	✗	✓	✓	✓
Hausdorff	✗	✗	✓	✓
Regular	✗	✗	✓	✓
Normal	✗	✗	✓	✓
Metrisable	✗	✗	✓	✓

Perfect map: continuous closed surjection with compact fibres. Domain assumed to have the property in each row.

The table also explains the use of invariants to separate spaces. The closed interval $[0,1]$ and the open interval $(0,1)$ are not homeomorphic because one is compact and the other is not; the circle S^1 and the interval $[0,1]$ are not homeomorphic because removing an interior point disconnects $[0,1]$ but not S^1. Conversely, the existence of a continuous surjection from $[0,1]$ onto the square $[0,1]^2$, the Peano curve, shows that dimension is not preserved by continuous surjections, although it is preserved by homeomorphisms by Brouwer's invariance of domain [4,6].

6. Discussion

Two lessons emerge from the systematic comparison. The first is that properties defined by covering conditions or by the absence of separations — compactness, the Lindelöf property, connectedness — travel forward along continuous maps, because continuity transports open covers and separations backward to the domain. Properties defined by the existence of many small open sets — the separation and countability axioms — do not, because a continuous map can coarsen the topology arbitrarily, as the indiscrete example shows.

The second lesson concerns hypotheses. Many errors in proofs come from assuming that a continuous bijection is a homeomorphism. Theorem 3 identifies the most useful sufficient condition, compact domain and Hausdorff codomain, and it underlies standard constructions such as the identification of $[0,1]$ with its endpoints glued to the circle S^1. Similarly, results about quotient spaces require care: quotients of Hausdorff spaces need not be Hausdorff, and conditions such as closedness of the equivalence relation must be checked separately [1,3].

These observations matter beyond pure topology. In functional analysis, continuity of linear operators and compactness of their images drive the theory of compact operators; in dynamical systems, topological conjugacy is a homeomorphism that intertwines two systems; in data analysis, persistent homology relies on the invariance of homology under homotopy equivalence [7]. In each case knowing exactly which properties a map preserves determines what can be inferred.

The study is restricted to classical point-set topology. Related notions — semi-continuity, weakly and strongly continuous functions, and the generalised continuities studied in recent literature on semi-open and pre-open sets [8,9] — follow similar patterns with weaker conclusions, and extending Table 1 to them is a natural line of future work.

7. Conclusions

Continuity is defined by a single condition on inverse images of open sets, yet it has many equivalent forms and very different consequences for different properties. This article has proved that compactness, connectedness, path-connectedness, separability and the Lindelöf property are preserved by continuous surjections; shown by counterexamples that the separation axioms, countability axioms and metrisability are not; and identified the stronger classes of maps — open, closed, perfect and homeomorphic — that do preserve them. The resulting preservation table is intended as a ready reference for teaching and for proofs that two spaces are, or are not, topologically equivalent.

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